Stability of closed characteristics on compact hypersurfaces in $\mathbf{R}^{2n}$ under pinching condition

Wei Wang*

School of Mathematical Science

Peking University, Beijing 100871

PEOPLES REPUBLIC OF CHINA

Jun. 22th, 2008

Abstract

In this article, let $\Sigma \subset \mathbf{R}^{2n}$ be a compact convex hypersurface which is (r, R)-pinched with $\frac{R}{r} < \sqrt{\frac{3}{2}}$. Then Σ carries at least two strictly elliptic closed characteristics; moreover, Σ carries at least $2[\frac{n+2}{4}]$ non-hyperbolic closed characteristics.

Key words: Compact convex hypersurfaces, closed characteristics, Hamiltonian systems, index iteration, stability.

AMS Subject Classification: 58E05, 37J45, 37C75.

Running title: Stability of closed characteristics

1 Introduction and main results

In this article, let Σ be a fixed C^3 compact convex hypersurface in $\mathbf{R}^{2n}$, i.e., Σ is the boundary of a compact and strictly convex region U in $\mathbf{R}^{2n}$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose U contains the origin. We consider closed characteristics (τ, y) on Σ, which are solutions of the following problem

$$\begin{cases} \dot{y} = JN_\Sigma(y), \\ y(\tau) = y(0), \end{cases} \tag{1.1}$$

*Partially supported by LMAM in Peking University in China and China Postdoctoral Science Foundation No.20070420264. E-mail: alexanderweiwang@yahoo.com.cn, wangwei@math.pku.edu.cn

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix in $\mathbf{R}^n$, $\tau > 0$ and $N_\Sigma(y)$ is the outward normal vector of Σ at y normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbf{R}^{2n}$. A closed characteristic (τ, y) is *prime* if τ is the minimal period of y. Two closed characteristics (τ, y) and (σ, z) are *geometrically distinct* if $y(\mathbf{R}) \neq z(\mathbf{R})$. We denote by $\mathcal{J}(\Sigma)$ and $\widetilde{\mathcal{J}}(\Sigma)$ the set of all closed characteristics (τ, y) on Σ with τ being the minimal period of y and the set of all geometrically distinct ones respectively. Note that $\mathcal{J}(\Sigma) = \{\theta \cdot y \,|\, \theta \in S^1, \ y \ is\ prime\}$, while $\widetilde{\mathcal{J}}(\Sigma) = \mathcal{J}(\Sigma)/S^1$, where the natural S^1-action is defined by $\theta \cdot y(t) = y(t + \tau\theta), \ \ \forall \theta \in S^1, \ t \in \mathbf{R}$.

Let $j : \mathbf{R}^{2n} \to \mathbf{R}$ be the gauge function of Σ, i.e., $j(\lambda x) = \lambda$ for $x \in \Sigma$ and $\lambda \geq 0$, then $j \in C^3(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^0(\mathbf{R}^{2n}, \mathbf{R})$ and $\Sigma = j^{-1}(1)$. Fix a constant $\alpha \in (1, +\infty)$ and define the Hamiltonian function $H_\alpha : \mathbf{R}^{2n} \to [0, +\infty)$ by

$$H_\alpha(x) = j(x)^\alpha, \qquad \forall x \in \mathbf{R}^{2n}. \tag{1.2}$$

Then $H_\alpha \in C^3(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. It is well known that problem (1.1) is equivalent to the following given energy problem of Hamiltonian system

$$\begin{cases} \dot{y}(t) = JH'_\alpha(y(t)), & H_\alpha(y(t)) = 1, \qquad \forall t \in \mathbf{R}. \\ y(\tau) = y(0). \end{cases} \tag{1.3}$$

Denote by $\mathcal{J}(\Sigma, \alpha)$ the set of all solutions (τ, y) of (1.3) where τ is the minimal period of y and by $\widetilde{\mathcal{J}}(\Sigma, \alpha)$ the set of all geometrically distinct solutions of (1.3). As above, $\widetilde{\mathcal{J}}(\Sigma, \alpha)$ is obtained from $\mathcal{J}(\Sigma, \alpha)$ by dividing the natural S^1-action. Note that elements in $\mathcal{J}(\Sigma)$ and $\mathcal{J}(\Sigma, \alpha)$ are one to one correspondent to each other, similarly for $\widetilde{\mathcal{J}}(\Sigma)$ and $\widetilde{\mathcal{J}}(\Sigma, \alpha)$.

Let $(\tau, y) \in \mathcal{J}(\Sigma, \alpha)$. The fundamental solution $\gamma_y : [0, \tau] \to \mathrm{Sp}(2n)$ with $\gamma_y(0) = I_{2n}$ of the linearized Hamiltonian system

$$\dot{w}(t) = JH''_\alpha(y(t))w(t), \qquad \forall t \in \mathbf{R}, \tag{1.4}$$

is called the *associate symplectic path* of (τ, y). The eigenvalues of $\gamma_y(\tau)$ are called *Floquet multipliers* of (τ, y). By Proposition 1.6.13 of [Eke3], the Floquet multipliers with their multiplicities of $(\tau, y) \in \mathcal{J}(\Sigma)$ do not depend on the particular choice of the Hamiltonian function in (1.3). For any $M \in \mathrm{Sp}(2n)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U} = \{z \in \mathbf{C} |\ |z| = 1\}$ in the complex plane $\mathbf{C}$. Since M is symplectic, $e(M)$ is even and $0 \leq e(M) \leq 2n$. As usual $(\tau, y) \in \mathcal{J}(\Sigma)$ is *elliptic* if $e(\gamma_y(\tau)) = 2n$. It is *strictly elliptic* if all the eigenvalues $\neq 1$ are Krein-definite. It is *non-degenerate* if 1 is a double

Floquet multiplier of it. It is *hyperbolic* if 1 is a double Floquet multiplier of it and $e(\gamma_y(\tau)) = 2$. It is well known that these concepts are independent of the choice of α.

As in Definition 5.1.7 of [Eke3], a C^3 hypersurface Σ bounding a compact convex region U, containing 0 in its interior is (r,R)-*pinched*, with $0 < r \le R$ if

$$|y|^2 R^{-2} \le \frac{1}{2} H_2''(x) y \cdot y \le |y|^2 r^{-2}, \qquad \forall x \in \Sigma, \quad \forall y \in \mathbf{R}^{2n}. \tag{1.5}$$

For the existence and multiplicity of geometrically distinct closed characteristics on convex compact hypersurfaces in $\mathbf{R}^{2n}$ we refer to [Rab1], [Wei1], [EkL1], [EkH1], [Szu1], [Vit1], [HWZ], [LoZ1], [LLZ], [WHL], and references therein.

On the stability problem, in [Eke2] of Ekeland in 1986 and [Lon2] of Long in 1998, for any $\Sigma \in \mathcal{H}(2n)$ the existence of at least one non-hyperbolic closed characteristic on Σ was proved provided $^{\#}\tilde{\mathcal{J}}(\Sigma) < +\infty$. Ekeland proved also in [Eke2] the existence of at least one strictly elliptic closed characteristic on Σ provided $\Sigma \in \mathcal{H}(2n)$ is $\sqrt{2}$-pinched. In [DDE1] of 1992, Dell'Antonio, D'Onofrio and Ekeland proved the existence of at least one elliptic closed characteristic on Σ provided $\Sigma = -\Sigma$. In [Lon4] of 2000, Long proved that $\Sigma \in \mathcal{H}(4)$ and $^{\#}\tilde{\mathcal{J}}(\Sigma) = 2$ imply that both of the closed characteristics must be elliptic. In [LoZ1] of 2002, Long and Zhu further proved when $^{\#}\tilde{\mathcal{J}}(\Sigma) < +\infty$, there exists at least one elliptic closed characteristic and there are at least $[\frac{n}{2}]$ geometrically distinct closed characteristics on Σ possessing irrational mean indices, which are then non-hyperbolic. In [LoW1], Long and the author proved that there exist at least two non-hyperbolic closed characteristics on $\Sigma \in \mathcal{H}(6)$ when $^{\#}\tilde{\mathcal{J}}(\Sigma) < +\infty$. In [W1], the author proved that on every $\Sigma \in \mathcal{H}(6)$ satisfying $^{\#}\tilde{\mathcal{J}}(\Sigma) < +\infty$, there exist at least two closed characteristics possessing irrational mean indices and if $^{\#}\tilde{\mathcal{J}}(\Sigma) = 3$, then there exist at least two elliptic closed characteristics. In [W2], the author studies stability of closed characteristics on symmetric hypersurfaces.

Motivated by these results, we prove the following results in this article:

Theorem 1.1. *let $\Sigma \subset \mathbf{R}^{2n}$ be a compact convex hypersurface which is (r,R)-pinched with $\frac{R}{r} < \sqrt{\frac{3}{2}}$. Then Σ carries at least two geometrically distinct strictly elliptic closed characteristics.*

Theorem 1.2. *let $\Sigma \subset \mathbf{R}^{2n}$ be a compact convex hypersurface which is (r,R)-pinched with $\frac{R}{r} < \sqrt{\frac{3}{2}}$. Then Σ carries at least $2[\frac{n+2}{4}]$ geometrically distinct non-hyperbolic closed characteristics.*

The proof of these theorems is motivated by the methods in [BTZ1], [Eke3] and [LoZ1] by using the index iteration theory and comparison theorems on indices as in the study of closed geodesics.

In this article, let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers and complex numbers respectively. Denote by $a \cdot b$ and $|a|$ the standard inner product and norm in $\mathbf{R}^{2n}$. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the standard

L^2-inner product and L^2-norm. For an S^1-space X, we denote by X_{S^1} the homotopy quotient of X module the S^1-action, i.e., $X_{S^1} = S^\infty \times_{S^1} X$. We define the functions

$$\begin{cases} [a] = \max\{k \in \mathbf{Z} \,|\, k \le a\}, \quad E(a) = \min\{k \in \mathbf{Z} \,|\, k \ge a\}, \\ \qquad \varphi(a) = E(a) - [a], \end{cases} \tag{1.6}$$

Specially, $\varphi(a) = 0$ if $a \in \mathbf{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbf{Z}$. In this article we use only $\mathbf{Q}$-coefficients for all homological modules.

2 Critical point theory for closed characteristics

In this section, we describe the critical point theory for closed characteristics.

As in P.199 of [Eke3], choose some $\alpha \in (1,\, 2)$ and associate with U a convex function H_α such that $H_\alpha(\lambda x) = \lambda^\alpha H_\alpha(x)$ for $\lambda \ge 0$. Consider the fixed period problem

$$\begin{cases} \dot{x}(t) = JH_\alpha'(x(t)), \\ \quad x(1) = x(0). \end{cases} \tag{2.1}$$

Define

$$L_0^{\frac{\alpha}{\alpha-1}}(S^1, \mathbf{R}^{2n}) = \{u \in L^{\frac{\alpha}{\alpha-1}}(S^1, \mathbf{R}^{2n}) \,|\, \int_0^1 u dt = 0\}. \tag{2.2}$$

The corresponding Clarke-Ekeland dual action functional is defined by

$$\Phi(u) = \int_0^1 \left(\frac{1}{2} Ju \cdot Mu + H_\alpha^*(-Ju)\right) dt, \qquad \forall\, u \in L_0^{\frac{\alpha}{\alpha-1}}(S^1, \mathbf{R}^{2n}), \tag{2.3}$$

where Mu is defined by $\frac{d}{dt} Mu(t) = u(t)$ and $\int_0^1 Mu(t) dt = 0$, H_α^* is the Fenchel transform of H_α defined by $H_\alpha^*(y) = \sup\{x \cdot y - H_\alpha(x) \mid x \in \mathbf{R}^{2n}\}$. By Theorem 5.2.8 of [Eke3], Φ is C^1 on $L_0^{\frac{\alpha}{\alpha-1}}$ and satisfies the Palais-Smale condition. Suppose x is a solution of (2.1). Then $u = \dot{x}$ is a critical point of Φ. Conversely, suppose u is a critical point of Φ. Then there exists a unique $\xi \in \mathbf{R}^{2n}$ such that $Mu - \xi$ is a solution of (2.1). In particular, solutions of (2.1) are in one to one correspondence with critical points of Φ. Moreover, $\Phi(u) < 0$ for every critical point $u \ne 0$ of Φ.

Suppose u is a nonzero critical point of Φ. Then the formal Hessian of Φ at u on $L_0^2(S^1, \mathbf{R}^{2n})$ is defined by

$$Q(v,\ v) = \int_0^1 (Jv \cdot Mv + (H_\alpha^*)''(-Ju) Jv \cdot Jv) dt,$$

which defines an orthogonal splitting $L_0^2(S^1, \mathbf{R}^{2n}) = E_- \oplus E_0 \oplus E_+$ of $L_0^2(S^1,\ \mathbf{R}^{2n})$ into negative, zero and positive subspaces. The index of u is defined by $i(u) = \dim E_-$ and the nullity of u is defined by $\nu(u) = \dim E_0$. Specially $1 \le \nu(u) \le 2n$ always holds, cf. P.219 of [Eke3].

We have a natural S^1-action on $L_0^{\frac{\alpha}{\alpha-1}}(S^1,\ \mathbf{R}^{2n})$ defined by $\theta \cdot u(t) = u(\theta + t)$ for all $\theta \in S^1$ and $t \in \mathbf{R}$. Clearly Φ is S^1-invariant. Hence if u is a critical point of Φ, then the whole orbit $S^1 \cdot u$ is

formed by critical points of Φ. Denote by $crit(\Phi)$ the set of critical points of Φ. Then $crit(\Phi)$ is compact by the Palais-Smale condition.

For a closed characteristic (τ, y) on Σ, we denote by $y^m \equiv (m\tau, y)$ the m-th iteration of y for $m \in \mathbf{N}$. Let u^m be the unique critical point of Φ corresponding to $(m\tau, y)$. Then we define the index $i(y^m)$ and nullity $\nu(y^m)$ of $(m\tau, y)$ for $m \in \mathbf{N}$ by

$$i(y^m) = i(u^m), \qquad \nu(y^m) = \nu(u^m). \tag{2.4}$$

The mean index of (τ, y) is defined by

$$\hat{i}(y) = \lim_{m\to\infty} \frac{i(y^m)}{m}. \tag{2.5}$$

Note that $\hat{i}(y) > 2$ always holds which was proved by Ekeland and Hofer in [EkH1] of 1987 (cf. Corollary 8.3.2 and Lemma 15.3.2 of [Lon5] for a different proof).

Recall that for a principal $U(1)$-bundle $E \to B$, the Fadell-Rabinowitz index (cf. [FaR1]) of E is defined to be $\sup\{k \mid c_1(E)^{k-1} \neq 0\}$, where $c_1(E) \in H^2(B, \mathbf{Q})$ is the first rational Chern class. For a $U(1)$-space, i.e., a topological space X with a $U(1)$-action, the Fadell-Rabinowitz index is defined to be the index of the bundle $X \times S^\infty \to X \times_{U(1)} S^\infty$, where $S^\infty \to CP^\infty$ is the universal $U(1)$-bundle.

For any $\kappa \in \mathbf{R}$, we denote by

$$\Phi^{\kappa-} = \{u \in L_0^{\frac{\alpha}{\alpha-1}}(S^1, \mathbf{R}^{2n}) \mid \Phi(u) < \kappa\}. \tag{2.6}$$

Then as in P.218 of [Eke3], we define

$$c_i = \inf\{\delta \in \mathbf{R} \mid \hat{I}(\Phi^{\delta-}) \geq i\}, \tag{2.7}$$

where $\hat{I}$ is the Fadell-Rabinowitz index defined above. Then by Proposition 5.3.3 in P.218 of [Eke3], we have

Proposition 2.1. *Each c_i is a critical value of Φ. If $c_i = c_j$ for some $i < j$, then there are infinitely many geometrically distinct closed characteristics on Σ.*

By Theorem 5.3.4 in P.219 of [Eke3], we have

Proposition 2.2. *For every $i \in \mathbf{N}$, there exists a point $u \in L_0^{\frac{\alpha}{\alpha-1}}(S^1, \mathbf{R}^{2n})$ such that*

$$\Phi'(u) = 0, \quad \Phi(u) = c_i, \tag{2.8}$$

$$i(u) \leq 2(i-1) \leq i(u) + \nu(u) - 1. \tag{2.9}$$

3 Index theory for closed characteristics

In this section, we recall briefly an index theory for symplectic paths developed by Y. Long and his coworkers. All the details can be found in [Lon5]. These results will be used in the next section.

As usual, the symplectic group $\mathrm{Sp}(2n)$ is defined by

$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(2n, \mathbf{R}) \mid M^T JM = J\},$$

whose topology is induced from that of $\mathbf{R}^{4n^2}$. For $\tau > 0$ we are interested in paths in $\mathrm{Sp}(2n)$:

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) \mid \gamma(0) = I_{2n}\},$$

which is equipped with the topology induced from that of $\mathrm{Sp}(2n)$. The following real function was introduced in [Lon3]:

$$D_\omega(M) = (-1)^{n-1}\overline{\omega}^n \det(M - \omega I_{2n}), \qquad \forall \omega \in \mathbf{U},\ M \in \mathrm{Sp}(2n).$$

Thus for any $\omega \in \mathbf{U}$ the following codimension 1 hypersurface in $\mathrm{Sp}(2n)$ is defined in [Lon3]:

$$\mathrm{Sp}(2n)^0_\omega = \{M \in \mathrm{Sp}(2n) \mid D_\omega(M) = 0\}.$$

For any $M \in \mathrm{Sp}(2n)^0_\omega$, we define a co-orientation of $\mathrm{Sp}(2n)^0_\omega$ at M by the positive direction $\frac{d}{dt}Me^{t\epsilon J}|_{t=0}$ of the path $Me^{t\epsilon J}$ with $0 \le t \le 1$ and $\epsilon > 0$ being sufficiently small. Let

$$\begin{aligned}
\mathrm{Sp}(2n)^*_\omega &= \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)^0_\omega, \\
\mathcal{P}^*_{\tau,\omega}(2n) &= \{\gamma \in \mathcal{P}_\tau(2n) \mid \gamma(\tau) \in \mathrm{Sp}(2n)^*_\omega\}, \\
\mathcal{P}^0_{\tau,\omega}(2n) &= \mathcal{P}_\tau(2n) \setminus \mathcal{P}^*_{\tau,\omega}(2n).
\end{aligned}$$

For any two continuous arcs ξ and $\eta : [0,\tau] \to \mathrm{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, it is defined as usual:

$$\eta * \xi(t) = \begin{cases} \xi(2t), & \text{if } 0 \le t \le \tau/2, \\ \eta(2t-\tau), & \text{if } \tau/2 \le t \le \tau. \end{cases}$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, as in [Lon5], the $\diamond$-product of M_1 and M_2 is defined by the following $2(m_1 + m_2) \times 2(m_1 + m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^{\diamond k}$ the k-fold $\diamond$-product $M\diamond\cdots\diamond M$. Note that the $\diamond$-product of any two symplectic matrices is symplectic. For any two paths $\gamma_j\in\mathcal{P}_\tau(2n_j)$ with $j=0$ and 1, let $\gamma_0\diamond\gamma_1(t)=\gamma_0(t)\diamond\gamma_1(t)$ for all $t\in[0,\tau]$.

A special path $\xi_n\in\mathcal{P}_\tau(2n)$ is defined by

$$\xi_n(t)=\begin{pmatrix}2-\frac{t}{\tau} & 0\\ 0 & (2-\frac{t}{\tau})^{-1}\end{pmatrix}^{\diamond n} \qquad \text{for} \quad 0\le t\le\tau. \tag{3.1}$$

Definition 3.1. (cf. [Lon3], [Lon5]) *For any $\omega\in\mathbf{U}$ and $M\in\mathrm{Sp}(2n)$, define*

$$\nu_\omega(M)=\dim_{\mathbf{C}}\ker_{\mathbf{C}}(M-\omega I_{2n}). \tag{3.2}$$

For any $\tau>0$ and $\gamma\in\mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma)=\nu_\omega(\gamma(\tau)). \tag{3.3}$$

If $\gamma\in\mathcal{P}^*_{\tau,\omega}(2n)$, define

$$i_\omega(\gamma)=[\mathrm{Sp}(2n)^0_\omega:\gamma*\xi_n], \tag{3.4}$$

where the right hand side of (3.4) is the usual homotopy intersection number, and the orientation of $\gamma*\xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma\in\mathcal{P}^0_{\tau,\omega}(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma)=\sup_{U\in\mathcal{F}(\gamma)}\inf\{i_\omega(\beta)\,|\,\beta\in U\cap\mathcal{P}^*_{\tau,\omega}(2n)\}. \tag{3.5}$$

Then

$$(i_\omega(\gamma),\nu_\omega(\gamma))\in\mathbf{Z}\times\{0,1,\ldots,2n\},$$

is called the index function of γ at ω.

Note that when $\omega=1$, this index theory was introduced by C. Conley-E. Zehnder in [CoZ1] for the non-degenerate case with $n\ge 2$, Y. Long-E. Zehnder in [LZe1] for the non-degenerate case with $n=1$, and Y. Long in [Lon1] and C. Viterbo in [Vit2] independently for the degenerate case. The case for general $\omega\in\mathbf{U}$ was defined by Y. Long in [Lon3] in order to study the index iteration theory (cf. [Lon5] for more details and references).

For any symplectic path $\gamma\in\mathcal{P}_\tau(2n)$ and $m\in\mathbf{N}$, we define its m-th iteration $\gamma^m:[0,m\tau]\to\mathrm{Sp}(2n)$ by

$$\gamma^m(t)=\gamma(t-j\tau)\gamma(\tau)^j, \qquad \text{for} \quad j\tau\le t\le(j+1)\tau,\ j=0,1,\ldots,m-1. \tag{3.6}$$

We still denote the extended path on $[0,+\infty)$ by γ.

Definition 3.2. (cf. [Lon3], [Lon5]) *For any $\gamma \in \mathcal{P}_\tau(2n)$, we define*

$$(i(\gamma, m), \nu(\gamma, m)) = (i_1(\gamma^m), \nu_1(\gamma^m)), \qquad \forall m \in \mathbf{N}. \tag{3.7}$$

The mean index $\hat{i}(\gamma, m)$ per $m\tau$ for $m \in \mathbf{N}$ is defined by

$$\hat{i}(\gamma, m) = \lim_{k \to +\infty} \frac{i(\gamma, mk)}{k}. \tag{3.8}$$

For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, the splitting numbers $S_M^{\pm}(\omega)$ of M at ω are defined by

$$S_M^{\pm}(\omega) = \lim_{\epsilon \to 0^+} i_{\omega \exp(\pm\sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma), \tag{3.9}$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$.

For a given path $\gamma \in \mathcal{P}_\tau(2n)$ we consider to deform it to a new path η in $\mathcal{P}_\tau(2n)$ so that

$$i_1(\gamma^m) = i_1(\eta^m), \quad \nu_1(\gamma^m) = \nu_1(\eta^m), \quad \forall m \in \mathbf{N}, \tag{3.10}$$

and that $(i_1(\eta^m), \nu_1(\eta^m))$ is easy enough to compute. This leads to finding homotopies $\delta : [0,1] \times [0,\tau] \to \mathrm{Sp}(2n)$ starting from γ in $\mathcal{P}_\tau(2n)$ and keeping the end points of the homotopy always stay in a certain suitably chosen maximal subset of $\mathrm{Sp}(2n)$ so that (3.10) always holds. In fact, this set was first discovered in [Lon3] as the path connected component $\Omega^0(M)$ containing $M = \gamma(\tau)$ of the set

$$\begin{aligned} \Omega(M) = \{N \in \mathrm{Sp}(2n) \quad | \quad & \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and} \\ & \nu_\lambda(N) = \nu_\lambda(M)\ \forall\, \lambda \in \sigma(M) \cap \mathbf{U}\}. \end{aligned} \tag{3.11}$$

Here $\Omega^0(M)$ is called the *homotopy component* of M in $\mathrm{Sp}(2n)$.

In [Lon3]-[Lon5], the following symplectic matrices were introduced as *basic normal forms*:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \lambda = \pm 2, \tag{3.12}$$

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \qquad \lambda = \pm 1, b = \pm 1, 0, \tag{3.13}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in (0,\pi) \cup (\pi, 2\pi), \tag{3.14}$$

$$N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \qquad \theta \in (0,\pi) \cup (\pi, 2\pi), \tag{3.15}$$

where $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbf{R}$ such that $(b_2 - b_3)\sin\theta > 0$, if $N_2(\omega, b)$ is trivial; $(b_2 - b_3)\sin\theta < 0$, if $N_2(\omega, b)$ is non-trivial.

Splitting numbers possess the following properties:

Lemma 3.3. (cf. [Lon3] and Lemmas 9.1.5 of [Lon5]) *Splitting numbers $S_M^{\pm}(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$ appeared in (3.9). For $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, splitting numbers $S_N^{\pm}(\omega)$ are constant for all $N \in \Omega^0(M)$.*

Lemma 3.4. (cf. [Lon3], Lemma 9.1.5-9.1.6 and List 9.1.12 of [Lon5]) *For $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, there hold*

$$S_M^+(\omega) = S_M^-(\overline{\omega}), \qquad \forall \omega \in \mathbf{U}. \tag{3.16}$$

$$S_M^{\pm}(\omega) = 0, \qquad \textit{if} \ \ \omega \notin \sigma(M). \tag{3.17}$$

$$(S_{N_1(1,a)}^+(1),\, S_{N_1(1,a)}^-(1)) = \begin{cases} (1,\,1), & \text{if} \ \ a \ge 0, \\ (0,\,0) & \text{if} \ \ a < 0. \end{cases} \tag{3.18}$$

$$(S_{N_1(-1,a)}^+(-1),\, S_{N_1(-1,a)}^-(-1)) = \begin{cases} (1,\,1), & \text{if} \ \ a \le 0, \\ (0,\,0) & \text{if} \ \ a > 0. \end{cases} \tag{3.19}$$

$$(S_{R(\theta)}^+(e^{\sqrt{-1}\theta}),\, S_{R(\theta)}^-(e^{\sqrt{-1}\theta})) = (0,\,1) \qquad \textit{if} \ \ \theta \in (0,\pi) \cup (\pi, 2\pi). \tag{3.20}$$

$$(S_{N_2(\omega,b)}^+(e^{\sqrt{-1}\theta}),\, S_{N_2(\omega,b)}^-(e^{\sqrt{-1}\theta})) = (1,\,1) \qquad \textit{if} \ \ (b_2 - b_3)\sin\theta < 0.. \tag{3.21}$$

$$(S_{N_2(\omega,b)}^+(e^{\sqrt{-1}\theta}),\, S_{N_2(\omega,b)}^-(e^{\sqrt{-1}\theta})) = (0,\,0) \qquad \textit{if} \ \ (b_2 - b_3)\sin\theta > 0.. \tag{3.22}$$

For any $M_i \in \mathrm{Sp}(2n_i)$ with $i = 0$ and 1, there holds

$$S_{M_0 \diamond M_1}^{\pm}(\omega) = S_{M_0}^{\pm}(\omega) + S_{M_1}^{\pm}(\omega), \qquad \forall\, \omega \in \mathbf{U}. \tag{3.23}$$

We have the following

Theorem 3.5. (cf. [Lon4] and Theorem 1.8.10 of [Lon5]) *For any $M \in \mathrm{Sp}(2n)$, there is a path $f : [0,1] \to \Omega^0(M)$ such that $f(0) = M$ and*

$$f(1) = M_1 \diamond \cdots \diamond M_k, \tag{3.24}$$

where each M_i is a basic normal form listed in (3.12)-(3.15) for $1 \le i \le k$. In particular, we have $e(f(1)) \le e(M)$.

Let $\Sigma \in \mathcal{H}(2n)$. Using notations in §1, for any $(\tau, y) \in \mathcal{J}(\Sigma, \alpha)$ and $m \in \mathbf{N}$, we define its m-th iteration $y^m : \mathbf{R}/(m\tau\mathbf{Z}) \to \mathbf{R}^{2n}$ by

$$y^m(t) = y(t - j\tau), \qquad \text{for} \quad j\tau \le t \le (j+1)\tau, \quad j = 0, 1, 2, \ldots, m-1. \tag{3.25}$$

We still denote by y its extension to $[0, +\infty)$.

We define via Definition 3.2 the following

$$
\begin{aligned}
S_y^{\pm}(\omega) &= S_{\gamma_y(\tau)}^{\pm}(\omega), && (3.26)\\
(i(y,m),\nu(y,m)) &= (i(\gamma_y,m),\nu(\gamma_y,m)), && (3.27)\\
\hat{i}(y,m) &= \hat{i}(\gamma_y,m), && (3.28)
\end{aligned}
$$

for all $m\in\mathbf{N}$, where γ_y is the associated symplectic path of (τ,y). Then we have the following

Theorem 3.6. (cf. Lemma 1.1 of [LoZ1], Theorem 15.1.1 of [Lon5]) *Suppose* $(\tau,y)\in\mathcal{J}(\Sigma,\alpha)$. *Then we have*

$$
i(y^m)\equiv i(m\tau,y)=i(y,m)-n,\quad \nu(y^m)\equiv\nu(m\tau,y)=\nu(y,m),\qquad \forall m\in\mathbf{N}, \tag{3.29}
$$

where $i(y^m)$ *and* $\nu(y^m)$ *are the index and nullity defined in §2. In particular, (2.5) and (3.8) coincide, thus we simply denote them by* $\hat{i}(y)$.

4 Proofs of the main theorems

In this section we give the proofs of the main theorems.

Suppose $(\tau,\,y)\in\mathcal{J}(\Sigma,\alpha)$. Then by Lemma 1.3 of [LoZ1] or Lemma 15.2.4 of [Lon5], there exist $P_y\in\mathrm{Sp}(2n)$ and $M_y\in\mathrm{Sp}(2n-2)$ such that

$$
\gamma_y(\tau)=P_y^{-1}(N_1(1,\,1)\diamond M_y)P_y, \tag{4.1}
$$

here we use notations in §3.

Since $H_2(\cdot)$ is positive homogeneous of degree-two, by (1.5) we have

$$
|x|^2R^{-2}\le H_2(x)\le |x|^2r^{-2},\qquad \forall x\in\Sigma. \tag{4.2}
$$

Recall that the action of a closed characteristic $(\tau,\,y)$ is defined by (cf. P190 of [Eke3])

$$
A(\tau,y)=\frac{1}{2}\int_0^{\tau}(Jy\cdot\dot{y})dt. \tag{4.3}
$$

Note that $A(\tau,y)$ is a geometric quantity depending only on how many times one runs around the closed characteristic. In fact, we have $A(\tau,y)=A(\sigma,y\circ\phi)$ for any orientation-preserving diffeomorphism $\phi:\mathbf{R}/\sigma\mathbf{Z}\to\mathbf{R}/\tau\mathbf{Z}$.

Comparing with the theorem of Morse-Schoenberg in the study of geodesics, we have the following

Lemma 4.1. *let $\Sigma \subset \mathbf{R}^{2n}$ be a compact convex hypersurface which is (r, R)-pinched. Suppose (τ, y) is a closed characteristic on Σ. Then we have the following*

$$A(\tau, y) > k\pi R^2 \Rightarrow i(y) \geq 2nk, \tag{4.4}$$

$$A(\tau, y) < k\pi r^2 \Rightarrow i(y) + \nu(y) \leq 2n(k-1) - 1. \tag{4.5}$$

Proof. By Proposition 1.7.5 of [Eke3], we have

$$i(y) = i_{T_\alpha}(x_\alpha), \qquad \forall \alpha \in (1,\, 2] \tag{4.6}$$

where $(T_\alpha,\, x_\alpha)$ is a solution of

$$\begin{cases} \dot{x}(t) = JH'_\alpha(x(t)), \qquad H_\alpha(x(t)) = 1, \qquad \forall t \in \mathbf{R}. \\ \quad x(T) = x(0) \end{cases} \tag{4.7}$$

and $i_{T_\alpha}(x_\alpha)$ is defined in §1.6 of [Eke3].

Now consider the following three Hamiltonian systems

$$\dot{y} = 2JR^{-2}y, \tag{4.8}$$

$$\dot{y} = 2JH''_2(x_2(t))y, \tag{4.9}$$

$$\dot{y} = 2Jr^{-2}y, \tag{4.10}$$

and the three corresponding quadratic forms on

$$L_0^2([0,\, s],\, \mathbf{R}^{2n}) = \{u \in L^2([0,\, s],\, \mathbf{R}^{2n}) \mid \int_0^1 udt = 0\}$$

$$Q_s^R(v,\ v) = \int_0^s \left(Jv \cdot Mv + \frac{R^2}{2}\|v\|^2\right) dt, \tag{4.11}$$

$$Q_s(v,\ v) = \int_0^s (Jv \cdot Mv + H''_2(x_2(t))^{-1} Jv \cdot Jv) dt, \tag{4.12}$$

$$Q_s^r(v,\ v) = \int_0^s \left(Jv \cdot Mv + \frac{r^2}{2}\|v\|^2\right) dt, \tag{4.13}$$

Note that by (1.5) we have $Q_s^R(v,\ v) \geq Q_s(v,\ v) \geq Q_s^r(v,\ v)$. Thus we have $i_s^R \leq i_s \leq i_s^r$, where i_s^R, i_s and i_s^r are the indices of Q_s^R, Q_s and Q_s^r.

Note that by (21) in P.191 of [Eke3], we have $A(\tau,\, y) = T_2$. Hence we have

$$i(y) = i_{T_2}(x_2) \geq i_{T_2}^R \geq 2nk, \tag{4.14}$$

where the last inequality follows by $T_2 = A(\tau, y) > k\pi R^2$ and Lemma 1.4.13 of [Eke3].

Denote by $L_0^2([0,\, T_2],\, \mathbf{R}^{2n}) = E_- \oplus E_0 \oplus E_+$ the orthogonal splitting of $L_0^2([0,\, T_2],\, \mathbf{R}^{2n})$ into negative, zero and positive subspaces. Then we have the following observation: If V is a subspace of $L_0^2([0,\, T_2],\, \mathbf{R}^{2n})$ such that Q_{T_2} is negative semi-definite, i.e., $\xi \in V$ implies $Q_{T_2}(\xi,\, \xi) \le 0$, then $\dim V \le \dim E_- + \dim E_0$. In fact, this is a simple fact of linear algebra: Let

$$pr_- : L_0^2([0,\, T_2],\, \mathbf{R}^{2n}) = E_- \oplus E_0 \oplus E_+ \to E_-$$

be the orthogonal projection. Consider $pr_-|V : V \to E_-$. Then $\xi \in \ker pr_-|V$ must belong to $E_0 \oplus E_+$. That is, since $Q_{T_2}(\xi,\, \xi) \le 0$, $\xi \in E_0$. From

$$\dim V = \dim(\mathrm{im} pr_-|V) + \dim(\ker pr_-|V)$$

we prove our claim.

Let $\epsilon > 0$ be small enough such that $A(\tau, y) < k\pi(r-\epsilon)^2$. If V is a subspace of $L_0^2([0,\, T_2],\, \mathbf{R}^{2n})$ such that $Q_{T_2}|V \le 0$, then $Q_{T_2}^{r-\epsilon}|V < 0$. Hence we have $\dim V \le i_{T_2}^{r-\epsilon}$. In particular, we have $\dim E_- + \dim E_0 \le i_{T_2}^{r-\epsilon}$. Hence by Lemma 1.4.13 of [Eke3], we have

$$\dim E_- + \dim E_0 \le i_{T_2}^{r-\epsilon} \le 2n(k-1). \tag{4.15}$$

Note that $\dim E_- = i(y)$ and $\dim E_0 = \nu(y) + 1$. Hence the lemma follows. ∎

Suppose $M \in Sp(2n)$ has the normal form decomposition

$$\begin{aligned} M = \quad & N_1(1,1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1,-1)^{\diamond p_+} \diamond N_1(-1,1)^{\diamond q_-} \diamond (-I_{2q_0}) \diamond N_1(-1,-1)^{\diamond q_+} \\ & \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \diamond N_2(\omega_1, u_1) \diamond \cdots \diamond N_2(\omega_{r_*}, u_{r_*}) \\ & \diamond N_2(\lambda_1, v_1) \diamond \cdots \diamond N_2(\lambda_{r_0}, v_{r_0}) \diamond M_0 \end{aligned} \tag{4.16}$$

where $N_2(\omega_j, u_j)$s are non-trivial and $N_2(\lambda_j, v_j)$s are trivial basic normal forms; $\sigma(M_0) \cap U = \emptyset$; p_-, p_0, p_+, q_-, q_0, q_+, r, r_* and r_0 are non-negative integers; $\omega_j = e^{\sqrt{-1}\alpha_j}$, $\lambda_j = e^{\sqrt{-1}\beta_j}$; θ_j, α_j, β_j $\in (0, \pi) \cup (\pi, 2\pi)$; these integers and real numbers are uniquely determined by M.

We have the following lemma concerning the iteration of indices.

Lemma 4.2. *Suppose $(\tau, y) \in \mathcal{J}(\Sigma, \alpha)$ such that $\gamma_y(\tau)$ can be deformed in $\Omega^0(\gamma_y(\tau))$ to M as in (4.16). Then we have $i(y,2) - 2i(y,1) \le n$ and $i(y,2) + \nu(y,2) - 2(i(y,1) + \nu(y,1)) \ge 1 - n$. In particular, we have the following*

(i) if $i(y,2) - 2i(y,1) = n$, then we have

$$M = N_1(1,1)^{\diamond p_-} \diamond I_{2p_0} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \tag{4.17}$$

with $p_- + p_0 + r = n$ and $\theta_k \in (\pi, 2\pi)$ for $1 \le k \le r$. In particular, (τ, y) is strictly elliptic.

(ii) if $i(y,2)+\nu(y,2)-2(i(y,1)+\nu(y,1))=1-n$, then we have

$$M=N_1(1,1)\diamond I_{2p_0}\diamond N_1(1,-1)^{\diamond p_+}\diamond R(\theta_1)\diamond\cdots\diamond R(\theta_r) \tag{4.18}$$

with $p_0+p_++r=n-1$ and $\theta_k\in(0,\pi)$ for $1\le k\le r$. In particular, (τ,y) is strictly elliptic.

Proof. By the Bott-type formulae, cf. Theorem 9.2.1 of [Lon5], we have

$$i(y,2)=i(\gamma_y,2)=i_1(\gamma_y)+i_{-1}(\gamma_y)=i(y,1)+i_{-1}(\gamma_y), \tag{4.19}$$

$$\nu(y,2)=\nu(\gamma_y,2)=\nu_1(\gamma_y)+\nu_{-1}(\gamma_y)=\nu(y,1)+\nu_{-1}(\gamma_y), \tag{4.20}$$

Hence we have $i(y,2)-2i(y,1)=i_{-1}(\gamma_y)-i_1(\gamma_y)$. By (3.9) we have

$$i_{-1}(\gamma_y)-i_1(\gamma_y)=S_M^+(1)+\sum_{0<\theta<\pi}(S_M^+(e^{\sqrt{-1}\theta})-S_M^-(e^{\sqrt{-1}\theta}))-S_M^-(-1). \tag{4.21}$$

Thus $i_{-1}(\gamma_y)-i_1(\gamma_y)\le n$ and (i) holds by Lemma 3.4.

Note that we have

$$\nu(y,2)-2\nu(y,1)=\nu_{-1}(\gamma_y)-\nu_1(\gamma_y). \tag{4.22}$$

Thus $i(y,2)+\nu(y,2)-2(i(y,1)+\nu(y,1))\ge 1-n$ and (ii) holds by (4.1) and Lemma 3.4. ∎

We have the following theorem due to I. Ekeland and J. Lasry.

Theorem 4.3. *Let $U\subset\mathbf{R}^{2n}$ be a convex compact set with non-empty interior, and let Σ be its boundary. Assume there is a point $x_0\in\mathbf{R}^{2n}$ such that*

$$r\le|x-x_0|\le R,\qquad\forall x\in\Sigma \tag{4.23}$$

and $\frac{R}{r}<\sqrt{2}$. Then Σ carries at least n geometrically distinct closed characteristics $\{(\tau_1,y_1),\ldots,(\tau_n,y_n)\}$ where τ_i is the minimal period of y_i, and the actions $A(\tau_i,y_i)$ satisfy:

$$\pi r^2\le A(\tau_i,y_i)\le\pi R^2,\qquad 1\le i\le n. \tag{4.24}$$

By the proof of the above theorem and Proposition 2.2, we have

Lemma 4.4. *Assume $\{(\tau_1,y_1),\ldots,(\tau_n,y_n)\}$ are the closed characteristics found in Theorem 4.3. Then we have*

$$\Phi'(u_i)=0,\quad \Phi(u_i)=c_i, \tag{4.25}$$

$$i(u_i)\le 2(i-1)\le i(u_i)+\nu(u_i)-1, \tag{4.26}$$

for $1\le i\le n$, where u_i is the unique critical point of Φ corresponding to (τ_i,y_i). ∎

Now we give the proofs of the main theorems.

Proof of Theorem 1.1. Suppose $\Sigma \subset \mathbf{R}^{2n}$ is a compact convex hypersurface which is (r, R)-pinched with $\frac{R}{r} < \sqrt{\frac{3}{2}}$. Then we have (1.5) and (4.2). From (4.2) we have

$$r \leq |x| \leq R, \qquad \forall x \in \Sigma \tag{4.27}$$

Thus by Theorem 4.3, we obtain n geometrically distinct prime closed characteristics $\{(\tau_1, y_1), \ldots, (\tau_n, y_n)\}$ such that (4.24)-(4.26) hold.

Claim. *The closed characteristics (τ_1, y_1) and (τ_n, y_n) must be strictly elliptic.*

Note that $i(y_1) = 0$ by (4.26). Thus by Theorem 5.1.10 of [Eke3], we have (τ_1, y_1) must be strictly elliptic. Here we can give another proof. By (4.24) and $A(2\tau_i, y_i) = 2A(\tau_i, y_i)$, we have

$$2\pi r^2 \leq A(2\tau_i, y_i) \leq 2\pi R^2, \qquad 1 \leq i \leq n. \tag{4.28}$$

Since $\frac{R}{r} < \sqrt{\frac{3}{2}}$, we have $A(2\tau_i, y_i) \geq 2\pi r^2 > \frac{4}{3}\pi R^2$. Thus by Lemma 4.1, we have

$$i(y_i^2) \geq 2n. \tag{4.29}$$

Hence by Theorem 3.6 we have

$$i(y_1, 2) - 2i(y_1, 1) = i(y_1^2) + n - 2(i(y_1) + n) \geq n. \tag{4.30}$$

Thus by (ii) of Lemma 4.2, we have (τ_1, y_1) is strictly elliptic.

Note that

$$i(y_n) \leq 2(n-1) \leq i(y_n) + \nu(y_n) - 1 \tag{4.31}$$

by (4.26). On the other hand, we have $A(2\tau_i, y_i) \leq 2\pi R^2 < 3\pi r^2$. Thus by Lemma 4.1, we have

$$i(y_i^2) + \nu(y_i^2) \leq 4n - 1. \tag{4.32}$$

Hence by Theorem 3.6 we have

$$\begin{aligned} & i(y_n, 2) + \nu(y_n, 2) - 2(i(y_n, 1) + \nu(y_n, 1)) \\ = \quad & i(y_n^2) + n - 2(i(y_n) + n) + \nu(y_n^2) - 2\nu(y_n) \leq 1 - n. \end{aligned} \tag{4.33}$$

Thus by (i) of Lemma 4.2, we have (τ_n, y_n) is strictly elliptic. ∎

Proof of Theorem 1.2. Suppose (τ, y) is a hyperbolic closed characteristic. Then we have

$$\gamma_y(\tau) = P_y^{-1}(N_1(1, 1)\diamond M_y)P_y, \tag{4.34}$$

with $\sigma(M_y)\cap \mathbf{U}=\emptyset$. Thus by Theorem 8.3.1 of [Lon5] and Theorem 3.6, we have

$$i(y^m)=m(i(y)+n+1)-n-1,\quad \nu(y^m)=1,\qquad \forall m\in \mathbf{N}. \tag{4.35}$$

Now suppose $\{(\tau_1,y_1),\ldots,(\tau_n,y_n)\}$ are the n geometrically distinct prime closed characteristics obtained in Theorem 4.3. Thus if (τ_i,y_i) is hyperbolic, we have $i(y_i)=2(i-1)$ by (4.26)

$$i(y_i^m)=m(i(y_i)+n+1)-n-1=2m(i-1)+(m-1)(n+1),\qquad \forall m\in \mathbf{N}. \tag{4.36}$$

Hence by (4.29) and (4.32), we have

$$2n\le i(y_i^2)=4(i-1)+n+1, \tag{4.37}$$

$$4(i-1)+n+1+1=i(y_i^2)+\nu(y_i^2)\le 4n-1. \tag{4.38}$$

Hence we have

$$n-1\le 4(i-1)\le 3(n-1). \tag{4.39}$$

Thus we have

$$E\left(\frac{n-1}{4}\right)\le (i-1)\le \left[\frac{3(n-1)}{4}\right]. \tag{4.40}$$

Hence there are at most $[\frac{3(n-1)}{4}]-E\left(\frac{n-1}{4}\right)+1$ hyperbolic closed characteristics in $\{(\tau_1,y_1),\ldots,(\tau_n,y_n)\}$. This implies that there are at least

$$n-\left[\frac{3(n-1)}{4}\right]+E\left(\frac{n-1}{4}\right)-1=2\left[\frac{n+2}{4}\right]$$

non-hyperbolic closed characteristics on Σ. ∎

References

[BTZ1] W. Ballmann, G. Thorbergsson and W. Ziller, Closed geodesics on positively curved manifolds. *Ann. of Math.* 116(1982), 213-247.

[BTZ2] W. Ballmann, G. Thorbergsson and W. Ziller, Existence of closed geodesics on positively curved manifolds. *J. Diff. Geod.* 18(1983), 221-252.

[CoZ1] C. Conley and E. Zehnder, Morse-type index theory for flows and periodic solutions for Hamiltonian equations. *Comm. Pure. Appl. Math.* 37 (1984) 207-253.

[DDE1] Dell'Antonio, G., B. D'Onofrio and I. Ekeland, Les systém hamiltoniens convexes et pairs ne sont pas ergodiques en general. *C. R. Acad. Sci. Paris.* Series I. 315 (1992), 1413-1415.

[Eke1] I. Ekeland, Une théorie de Morse pour les systèmes hamiltoniens convexes. *Ann. IHP. Anal. non Linéaire.* 1 (1984) 19-78.

[Eke2] I. Ekeland, An index throry for periodic solutions of convex Hamiltonian systems. *Proc. Symp. in Pure Math.* 45 (1986) 395-423.

[Eke3] I. Ekeland, Convexity Methods in Hamiltonian Mechanics. Springer-Verlag. Berlin. 1990.

[EkH1] I. Ekeland and H. Hofer, Convex Hamiltonian energy surfaces and their closed trajectories. *Comm. Math. Phys.* 113 (1987) 419-467.

[EL] I. Ekeland and J. Lasry, On the number of periodic trajectories for a Hamiltonian flow on a convex energy surface. *Ann. of Math.* 112 (1980), 283–319.

[EkL1] I. Ekeland and L. Lassoued, Multiplicité des trajectoires fermées d'un systéme hamiltonien sur une hypersurface d'energie convexe. *Ann. IHP. Anal. non Linéaire.* 4 (1987) 1-29.

[FaR1] E. Fadell and P. Rabinowitz, Generalized comological index throries for Lie group actions with an application to bifurcation equations for Hamiltonian systems.

[HWZ] H. Hofer, K. Wysocki and E. Zehnder, The dynamics on three-dimensional strictly convex energy surfaces. *Ann. of Math.* 148 (1998) 197-289.

[LLZ] C. Liu, Y. Long and C. Zhu, Multiplicity of closed characteristics on symmetric convex hypersurfaces in $\mathbf{R}^{2n}$. *Math. Ann.* 323 (2002), 201-215.

[Lon1] Y. Long, Maslov-type index, degenerate critical points and asymptotically linear Hamiltonian systems. *Science in China.* Series A. 33(1990), 1409-1419.

[Lon2] Y. Long, Hyperbolic closed characteristics on compact convex smooth hypersurfaces in $\mathbf{R}^{2n}$. *J. Diff. Equa.* 150 (1998), 227-249.

[Lon3] Y. Long, Bott formula of the Maslov-type index theory. *Pacific J. Math.* 187 (1999), 113-149.

[Lon4] Y. Long, Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics. *Advances in Math.* 154 (2000), 76-131.

[Lon5] Y. Long, Index Theory for Symplectic Paths with Applications. Progress in Math. 207, Birkhäuser. Basel. 2002.

[Lon6] Y. Long, Index iteration theory for symplectic paths with applications to nonlinear Hamiltonian systems. *Proc. of Inter. Congress of Math. 2002.* Vol.II, 303-313. Higher Edu. Press. Beijing. 2002.

[LoW1] Y. Long and W. Wang, Stability of closed characteristics on compact convex hypersurfaces, *Memory Volume for Professor S. S. Chern.* Ed. by P. Griffiths. Nankai Tracts in Mathematics Vol. 11, World Scientific. 313-333..

[LZe1] Y. Long and E. Zehnder, Morse theory for forced oscillations of asymptotically linear Hamiltonian systems. In *Stoc. Proc. Phys. and Geom.*, S. Albeverio et al. ed. World Sci. (1990) 528-563.

[LoZ1] Y. Long and C. Zhu, Closed characteristics on compact convex hypersurfaces in $\mathbf{R}^{2n}$. *Ann. of Math.* 155 (2002) 317-368.

[Rab1] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems. *Comm. Pure Appl. Math.* 31 (1978) 157-184.

[Szu1] A. Szulkin, Morse theory and existence of periodic solutions of convex Hamiltonian systems. *Bull. Soc. Math. France.* 116 (1988) 171-197.

[Vit1] C. Viterbo, Equivariant Morse theory for starshaped Hamiltonian systems. *Trans. Amer. Math. Soc.* 311 (1989) 621-655.

[Vit2] C. Viterbo, A new obstruction to embedding Lagrangian tori. *Invent. Math.* 100 (1990) 301-320.

[WHL] W. Wang, X. Hu and Y. Long, Resonance identity, stability and multiplicity of closed characteristics on compact convex hypersurfaces. *Duke Math. J.* Volume 139, Number 3 (2007), 411-462.

[W1] W. Wang, Stability of closed characteristics on compact convex hypersurfaces in R^6. math.SG/0701673. *to appear in J. Eur. Math. Soc.*

[W2] W. Wang, Stability of closed characteristics on symmetric compact convex hypersurfaces in $\mathbf{R}^{2n}$. *preprint*

[Wei1] A. Weinstein, Periodic orbits for convex Hamiltonian systems. *Ann. of Math.* 108 (1978) 507-518.